

Makespan Investigations of Sequential, Parallel, PO, and POCL Plans

Harrison Oates Pascal Bercher
School of Computing, Australian National University

Introduction

Non-sequential plan representations, especially **partial order causal link (POCL)** plans, are useful for modelling concurrency, but their computational properties are often misunderstood.

Makespan is the minimum time required to execute a plan under parallelism.

The figures below show three plans for making breakfast. Figure 1 is a parallel plan, which comprises of action sets, Figure 2 is a partial order (PO) plan, and Figure 3 is a POCL plan. Each of these solve the breakfast problem with **makespan 2**.

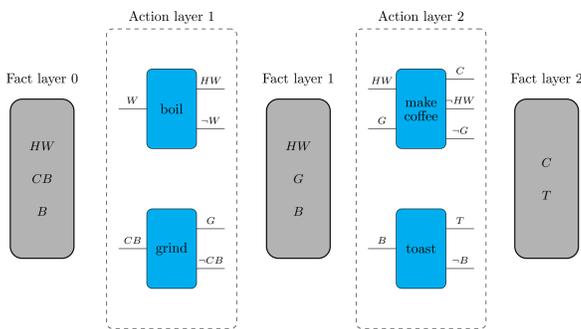


Figure 1: Parallel plan

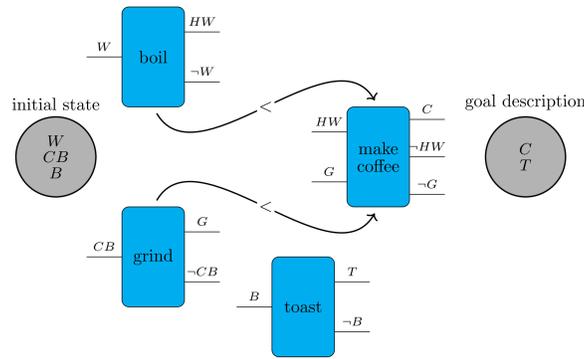


Figure 2: PO plan

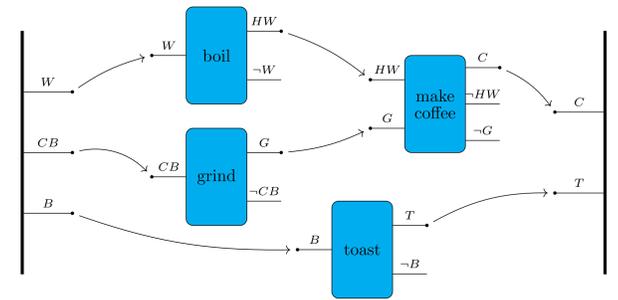


Figure 3: POCL plan

Makespan Convertibility Does Not Hold

In the ordering hierarchy:

Sequential \rightarrow Parallel \rightarrow PO \rightarrow POCL

Key Finding: Makespan can be preserved going from **left to right**, but **NOT** from right to left (except POCL \rightarrow PO).

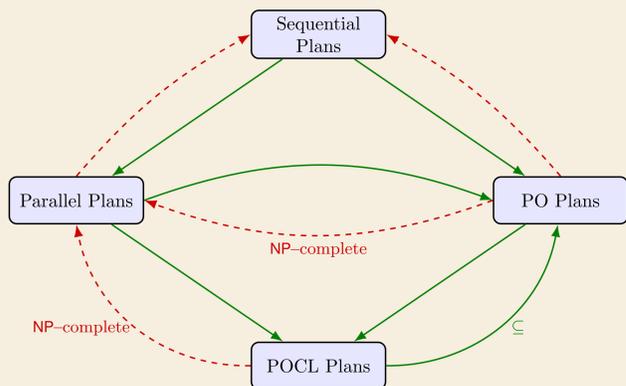


Figure 4: Makespan convertibility. Solid green: preserved. Dashed red: NOT preserved.

The Counterexample

In the below POCL plan, while a_1 and a_2 are unordered and thus share the same optimal release time, they have inconsistent effects, which violates the non-interference criteria for parallel plans.

Therefore, a_1 and a_2 **must** be serialized in any valid parallel plan, even though the POCL structure allows them to be concurrent!

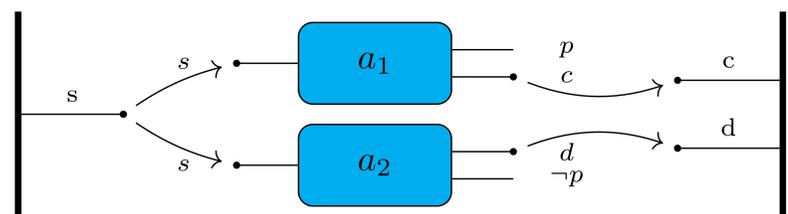


Figure 5: POCL plan with inconsistent effects

POCL to Parallel Conversion: Complexity & Bounds

Theorem: It is **NP-complete** to transform a POCL plan into a makespan-optimal parallel plan.

Proof Strategy:

- **Membership:** Guess partition, verify in polynomial time
- **Hardness:** Reduction from graph k -colouring
- Build conflict graph G_t for each timestep t
- Vertices = plan steps, edges = interfering actions
- Chromatic number $\chi(G_t)$ determines layers needed

Makespan Bounds:

Given POCL plan with makespan k and C interfering pairs:

- **Upper bound (any coloring):** $m \leq k + C$
- **Upper bound (optimal coloring):** $m \in O\left(k\sqrt{1 + \frac{C}{k}}\right)$
- **Worst case:** Sequential execution (no parallelism)

Consequences & Impact

1. We refute **Theorem 1** of Pecora et al. (2006), which claimed planning graph planners maximise PO concurrency.
2. **Heuristics admissible for parallel planning are inadmissible for PO/POCL.** Example: h_p^m (Haslum & Geffner, 2000). Since h_p^2 corresponds to the Graphplan heuristic h_G , and our counterexample achieves both goals in 1 timestep while $h_G(\{c, d\}) = 2$, the heuristic overestimates.
3. 'Optimal' POCL planners like CPT (Vidal & Geffner, 2006) guarantee optimality *only within the space representable as parallel plans*, not within the general POCL space.

Makespan-Bounded Plan Existence

Problem: Given planning problem Π and bound k , does Π have a plan with makespan $\leq k$?

- **PSPACE-complete** when k encoded in **binary**
Proof: Adapt Graphplan with polynomial space; reduce from bounded sequential plan existence
- **NP-complete** when k encoded in **unary**
Proof: Same reduction; unary k means $|\langle k \rangle| = k$, so time is polynomial in input size

These results apply uniformly to parallel, PO, and POCL plans!



Australian
National
University